Vector spherical harmonics

In mathematics, **vector spherical harmonics (VSH)** are an extension of the scalar spherical harmonics for the use with vector fields.

Definition

Several conventions have been used to define the VSH[1][2][3][4][5]. We follow that of Barrera *et al.*. Given a scalar spherical harmonic $Y_{lm}(\theta,\varphi)$ we define three VSH:

- $\mathbf{Y}_{lm} = Y_{lm}\hat{\mathbf{r}}$
- $\mathbf{\Psi}_{lm} = r\nabla Y_{lm}$
- $\mathbf{\Phi}_{lm} = \vec{\mathbf{r}}\times\nabla Y_{lm}$

being $\hat{\mathbf{r}}$ the unitary vector along the radial direction and $\vec{\mathbf{r}}$ the position vector of the point with spherical coordinates r , θ and ϕ . The radial factors are included to guarantee that the dimensions of the VSH are the same as the ordinary spherical harmonics and that the VSH do not depend on the radial spherical coordinate.

The interest of these new vector fields is to separate the radial dependence from the angular one when using spherical coordinates, so that a vector field admits a multipole expansion

$$\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(E^r_{lm}(r)\mathbf{Y}_{lm} + E^{(1)}_{lm}(r)\mathbf{\Psi}_{lm} + E^{(2)}_{lm}(r)\mathbf{\Phi}_{lm}\right)$$

The labels on the components reflect that E^r_{lm} is the radial component of the vector field, while $E^{(1)}_{lm}$ and $E^{(2)}_{lm}$ are transverse components.

Main Properties

Symmetry

Like the scalar spherical harmonics, the VSH satisfy

$$\mathbf{Y}_{l,-m} = (-1)^m\mathbf{Y}^*_{lm} \qquad \mathbf{\Psi}_{l,-m} = (-1)^m\mathbf{\Psi}^*_{lm} \qquad \mathbf{\Phi}_{l,-m} = (-1)^m\mathbf{\Phi}^*_{lm}$$

Orthogonality

The VSH are orthogonal in the usual three-dimensional way

$$\mathbf{Y}_{lm}\cdot\mathbf{\Psi}_{lm} = 0 \qquad \mathbf{Y}_{lm}\cdot\mathbf{\Phi}_{lm} = 0 \qquad \mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}_{lm} = 0$$

but also in the Hilbert space

$$\int\mathbf{Y}_{lm}\cdot\mathbf{Y}^*_{l'm'}\,\mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$

$$\int\mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}\,\mathrm{d}\Omega = l(l+1)\delta_{ll'}\delta_{mm'}$$

$$\int\mathbf{\Phi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}\,\mathrm{d}\Omega = l(l+1)\delta_{ll'}\delta_{mm'}$$

$$\int\mathbf{Y}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}\,\mathrm{d}\Omega = 0$$

$$\int\mathbf{Y}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}\,\mathrm{d}\Omega = 0$$

$$\int\mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}\,\mathrm{d}\Omega = 0$$

Vector multipole moments

The orthogonality relations allow to compute the spherical multipole moments of a vector field as

$$E^r_{lm} = \int \mathbf{E}\cdot\mathbf{Y}^*_{lm}\,\mathrm{d}\Omega$$
$$E^{(1)}_{lm} = \frac{1}{l(l+1)}\int \mathbf{E}\cdot\mathbf{\Psi}^*_{lm}\,\mathrm{d}\Omega$$
$$E^{(2)}_{lm} = \frac{1}{l(l+1)}\int \mathbf{E}\cdot\mathbf{\Phi}^*_{lm}\,\mathrm{d}\Omega$$

The gradient of a scalar field

Given the multipole expansion of a scalar field

$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\phi_{lm}(r)Y_{lm}(\theta,\phi)$$

we can express its gradient in terms of the VSH as

$$\nabla\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{\mathrm{d}\phi_{lm}}{\mathrm{d}r}\mathbf{Y}_{lm}+\frac{\phi_{lm}}{r}\mathbf{\Psi}_{lm}\right)$$

Divergence

For any multipole field we have

$$\nabla\cdot(f(r)\mathbf{Y}_{lm}) = \left(\frac{\mathrm{d}f}{\mathrm{d}r}+\frac{2}{r}f\right)Y_{lm}$$
$$\nabla\cdot(f(r)\mathbf{\Psi}_{lm}) = -\frac{l(l+1)}{r}fY_{lm}$$
$$\nabla\cdot(f(r)\mathbf{\Phi}_{lm}) = 0$$

By superposition we obtain the divergence of any vector field

$$\nabla\cdot\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{\mathrm{d}E^r_{lm}}{\mathrm{d}r}+\frac{2}{r}E^r_{lm}-\frac{l(l+1)}{r}E^{(1)}_{lm}\right)Y_{lm}$$

we see that the component on $\mathbf{\Phi}_{lm}$ is always solenoidal.

Curl

For any multipole field we have

$$\nabla\times(f(r)\mathbf{Y}_{lm}) = -\frac{1}{r}f\mathbf{\Phi}_{lm}$$
$$\nabla\times(f(r)\mathbf{\Psi}_{lm}) = \left(\frac{\mathrm{d}f}{\mathrm{d}r}+\frac{1}{r}f\right)\mathbf{\Phi}_{lm}$$
$$\nabla\times(f(r)\mathbf{\Phi}_{lm}) = -\frac{l(l+1)}{r}f\mathbf{Y}_{lm}-\left(\frac{\mathrm{d}f}{\mathrm{d}r}+\frac{1}{r}f\right)\mathbf{\Psi}_{lm}$$

By superposition we obtain the curl of any vector field

$$\nabla\times\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(-\frac{l(l+1)}{r}E^{(2)}_{lm}\mathbf{Y}_{lm}-\left(\frac{\mathrm{d}E^{(2)}_{lm}}{\mathrm{d}r}+\frac{1}{r}E^{(2)}_{lm}\right)\mathbf{\Psi}_{lm}+\left(-\frac{1}{r}E^r_{lm}+\frac{\mathrm{d}E^{(1)}_{lm}}{\mathrm{d}r}+\frac{1}{r}E^{(1)}_{lm}\right)\mathbf{\Phi}_{lm}\right)$$

Examples

First vector spherical harmonics

- $l = 0$
  - $\mathbf{Y}_{00} = \sqrt{\frac{1}{4\pi}}\hat{\mathbf{r}}$
  - $\mathbf{\Psi}_{00} = \mathbf{0}$
  - $\mathbf{\Phi}_{00} = \mathbf{0}$
- $l = 1$
  - $\mathbf{Y}_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta\,\hat{\mathbf{r}}$
  - $\mathbf{Y}_{11} = -\sqrt{\frac{3}{8\pi}}\mathrm{e}^{\mathrm{i}\varphi}\sin\theta\,\hat{\mathbf{r}}$
  - $\mathbf{\Psi}_{10} = -\sqrt{\frac{3}{4\pi}}\sin\theta\,\hat{\theta}$
  - $\mathbf{\Psi}_{11} = -\sqrt{\frac{3}{8\pi}}\mathrm{e}^{\mathrm{i}\varphi}\left(\cos\theta\,\hat{\theta} + \mathrm{i}\,\hat{\varphi}\right)$
  - $\mathbf{\Phi}_{10} = -\sqrt{\frac{3}{4\pi}}\sin\theta\,\hat{\varphi}$
  - $\mathbf{\Phi}_{11} = \sqrt{\frac{3}{8\pi}}\mathrm{e}^{\mathrm{i}\varphi}\left(\mathrm{i}\,\hat{\theta} - \cos\theta\,\hat{\varphi}\right)$

The expression for negative values of m are obtained applying the symmetry relations.

Application to electrodynamics

The VSH are especially useful in the study of multipole radiation fields. For instance, a magnetic multipole is due to an oscillating current with angular frequency ω and complex amplitude

$$\hat{\mathbf{J}} = J(r)\mathbf{\Phi}_{lm}$$

and the corresponding electric and magnetic fields can be written as

$$\hat{\mathbf{E}} = E(r)\mathbf{\Phi}_{lm}$$
$$\hat{\mathbf{B}} = B^r(r)\mathbf{Y}_{lm} + B^{(1)}(r)\mathbf{\Psi}_{lm}$$

Substituting into Maxwell equations, Gauss' law is automatically satisfied

$$\nabla \cdot \hat{\mathbf{E}} = 0$$

while Faraday's law decouples in

$$\nabla \times \hat{\mathbf{E}} = -\mathrm{i}\omega\hat{\mathbf{B}} \qquad \Rightarrow \qquad \begin{cases} \dfrac{l(l+1)}{r}E = \mathrm{i}\omega B^r \\ \dfrac{\mathrm{d}E}{\mathrm{d}r} + \dfrac{E}{r} = \mathrm{i}\omega B^{(1)} \end{cases}$$

Gauss' law for the magnetic field implies

$$\nabla \cdot \hat{\mathbf{B}} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}B^r}{\mathrm{d}r} + \frac{2}{r}B^r - \frac{l(l+1)}{r}B^{(1)} = 0$$

and Ampère-Maxwell's equation gives

$$\nabla \times \hat{\mathbf{B}} = \mu_0 \hat{\mathbf{J}} + \mathrm{i}\mu_0\varepsilon_0\omega\hat{\mathbf{E}} \quad \Rightarrow \quad -\frac{B^r}{r} + \frac{\mathrm{d}B^{(1)}}{\mathrm{d}r} + \frac{B^{(1)}}{r} = \mu_0 J + \mathrm{i}\omega\mu_0\varepsilon_0 E$$

In this way, the partial differential equations have been transformed in a set of ordinary differential equations.

Application to fluid dynamics

In the calculation of the Stokes' law for the drag that a viscous fluid exerts on a small spherical particle, the velocity distribution obeys Navier-Stokes equations neglecting inertia, i.e.

$$\nabla \cdot \mathbf{v} = 0$$

$$\mathbf{0} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

with the boundary conditions

$$\mathbf{v} = \mathbf{0} \quad (r = a)$$

$$\mathbf{v} = -\mathbf{U}_0 \quad (r \to \infty)$$

being $\mathbf{U}$ the relative velocity of the particle to the fluid far from the particle. In spherical coordinates this velocity at infinity can be written as

$$\mathbf{U}_0 = U_0 \left(\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\theta}\right) = U_0 \left(\mathbf{Y}_{10} + \mathbf{\Psi}_{10}\right)$$

The last expression suggest an expansion on spherical harmonics for the liquid velocity and the pressure

$$p = p(r) Y_{10}$$

$$\mathbf{v} = v^r(r)\mathbf{Y}_{10} + v^{(1)}(r)\mathbf{\Psi}_{10}$$

Substitution in the Navier-Stokes equations produces a set of ordinary differential equations for the coefficients.

External links

- *Vector Spherical Harmonics* at Eric Weisstein's Mathworld [6]

References

[1] R.G. Barrera, G.A. Estévez and J. Giraldo, *Vector spherical harmonics and their application to magnetostatics*, Eur. J. Phys. **6** 287-294 (1985)

[2] B. Carrascal, G.A. Estevez, P. Lee and V. Lorenzo *Vector spherical harmonics and their application to classical electrodynamics*, Eur. J. Phys., **12**, 184-191 (1991)

[3] E. L. Hill, *The theory of Vector Spherical Harmonics*, Am. J. Phys. **22**, 211-214 (1954)

[4] E. J. Weinberg, *Monopole vector spherical harmonics*, Phys. Rev. D. **49**, 1086-1092 (1994)

[5] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II*, New York: McGraw-Hill, 1898-1901 (1953)

[6] http://mathworld.wolfram.com/VectorSphericalHarmonic.html

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